Motivic classes of varieties and stacks with applications to Higgs bundles

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Overview

Introduction

Our primary goal is to give an introduction to motivic classes of varieties and stacks with applications to moduli of Higgs bundles.

Sources of Motivation:

- The work of Hausel, Letellier, Rodriguez-Villegas, and others regarding mixed Hodge polynomials of character varieties.
- Point counting for algebraic varieties and stacks over a finite field $\mathbb{F}_q$. Specifically, computations for moduli stacks of Higgs bundles done by Mozgovoy, Schiffmann, and Mellit.

Counting points of varieties over a finite field

Example

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field $\mathbb{F}_q$ with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathrm{GL}(n)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$ considered as the space of n linearly independent columns.

Motivic classes of varieties

Preliminary definition

For any field k, one defines the abelian group $\mathrm{Mot}_{var}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:

$$[X] = [Y] + [X - Y] \text{ where } Y \text{ is a closed subvariety of } X.$$

The class $[X]$ in $\mathrm{Mot}_{var}(k)$ is called the motivic class of the variety X. We can give a ring structure for $\mathrm{Mot}_{var}(k)$ by $[X] \cdot [Y] := [(X \times_k Y)_{red}]$ with the unit element $[\mathrm{Spec}\, k] = 1$.

Note that $[\emptyset] = 0$. Similarly we can define the motivic classes of schemes of finite type over k,by $[X] := [X_{red}]$.

Realizations of the motivic classes

Lemma

Let A be an abelian group and f be an A-valued function on isomorphism classes of k-varieties such that for all closed subvarieties $Y \subset X$, we have $f(X) = f(Y) + f(X - Y)$. Then there is a unique $\tilde{f} : \mathrm{Mot}_{var}(k) \to A$ such that for all varieties X we have $\tilde{f}([X]) = f(X)$.

Example

- Point counting: For $k = \mathbb{F}_q$, define $\# : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$ as $\#([X]) = |X(\mathbb{F}_q)|$, the number of rational points over $\mathbb{F}_q$.
- Euler characteristic: For $k = \mathbb{C}$, define $\chi : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$ such that $\chi([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q})$ is the Euler characteristic of X.
- E-polynomial: For $k = \mathbb{C}$, define $E : \mathrm{Mot}_{var}(k) \to \mathbb{Z}[u, v]$ by setting $E([X]) = \sum_{k,p,q} (-1)^k h^{k,p,q} u^p v^q$ such that $h^{k,p,q} = \dim_{\mathbb{C}} H_c^{k,p,q}(X)$ for any quasi-projective variety X.

Examples

So, instead of performing computations of invariants directly, we can instead perform computations of motivic classes in $\mathrm{Mot}_{var}(k)$. Letting $[\mathbb{A}^1_k] = \mathbb{L}$, we see that the same methods used for counting points over $\mathbb{F}_q$ give us:

Example

- $[\mathbb{A}^n] = \mathbb{L}^n$.
- $[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$.
- $[\mathrm{GL}(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$.

Local zeta function

Definition

The local zeta function of a variety X is defined as

$$Z(X,t) = \exp\left(\sum_{k=1}^{\infty} \frac{|X(\mathbb{F}_{q^k})|}{k} t^k\right)$$

We prefer to rewrite local zeta function in terms of symmetric products.

Proposition

$$Z(X,t) = \sum_{n\geq 0} |Sym^n X(\mathbb{F}_q)| \cdot t^n$$

where $Sym^n X$ is the symmetric product X^n/S_n (with the convention that $Sym^0 X = \operatorname{Spec} k$).

Compare the two versions of local zeta function

We verify the above proposition for the coefficient of t^2.

Example

For the original local zeta function, we have two cases $n = 1$ and $n = 2$ as follows:

$$\frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

For the symmetric product $|Sym^2 X(\mathbb{F}_q)|$, we have the following two cases. If $x, y \in \mathbb{F}_q$, the coefficient is $|Sym^2(X(\mathbb{F}_q))|$. If $x \in \mathbb{F}_{q^2} - \mathbb{F}_q$, the possible type is $(x, \phi(x))$, where ϕ is a Frobenius morphism. Finally we have

$$|Sym^2(X(\mathbb{F}_q))| + \frac{|X(\mathbb{F}_{q^2})| - |X(\mathbb{F}_q)|}{2} = \frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

Motivic zeta function

Definition

We can define the motivic zeta function for every quasi-projective variety X:

$$Z_X(t) = \sum_{n \geq 0} [Sym^n X] \cdot t^n \in 1 + t \cdot \mathrm{Mot}_{var}(k)[[t]].$$

- If $k = \mathbb{F}_q$, then the image of this series under $\# : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$ coincides with the local zeta function.

Counting points of projective spaces

We are interested in performing computations for quotients. One of the motivation comes from the following computations for projective spaces.

Example

- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.

- $|\mathbb{P}^n|$ can also be computed by presenting $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathrm{GL}(1)$.

$$\left| \frac{\mathbb{A}^{n+1} - \{0\}}{\mathrm{GL}(1)} \right| = \frac{q^{n+1} - 1}{q - 1} = 1 + q + \cdots + q^n$$

Counting points of groupoids

If $\mathcal{G}$ is a groupoid, we can define the volume of $\mathcal{G}$ as a weighted sum:

$$|\mathcal{G}| := \sum_{G \in \mathcal{G}} \frac{1}{|\operatorname{Aut}(G)|}.$$

Example (Funny)

The volume of the groupoid of finite sets is e. Indeed, for any finite set of size n, the automorphism group is of order $n!$:

$$|\mathcal{G}| = \sum_{G \in \mathcal{G}} \frac{1}{|\operatorname{Aut}(G)|} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Counting points of algebraic stacks over a finite field

- We can try to do similar computations even when the group action is not free and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If $\mathcal{Y}$ is an algebraic stack over $\mathbb{F}_q$, we can generalize point counting for varieties by defining the volume (also known as the mass) of $\mathcal{Y}$ as a weighted sum over all its $\mathbb{F}_q$-rational points:

$$|\mathcal{Y}| := \sum_{y \in \mathrm{Ob}(\mathcal{Y}(\mathbb{F}_q))} \frac{1}{|\mathrm{Aut}(y)|}.$$

This is not guaranteed to converge. However, all the stacks we consider are of finite type, for which it converges (the sum is finite).

Example: volume of the stack of vector bundles

We give an example where the stack is not of finite type but the volume converges.

Siegel formula

Let X be a smooth projective curve of genus g over $\mathbb{F}_q$. Let $\mathcal{B}un_{r,d}(X)$ be the moduli stack parametrizing isomorphism classes of rank r, degree d vector bundles over X. The volume over $\mathbb{F}_q$ may be computed as:

$$|\mathcal{B}un_{r,d}(X)| = \frac{q^{(r^2-1)(g-1)}}{q-1}|Jac(X)|\prod_{i=2}^{r} Z_X(q^{-i})$$

where $Jac(X)$ is the Jacobian of X, i.e. the moduli space of degree 0 line bundles on X.

Motivic classes of stacks

We use the convention that all the stacks considered will be Artin stacks locally of finite type over a field k such that the stablizers of points are affine.

Definition

One defines the abelian group $\mathrm{Mot}(k)$ as the group generated by isomorphism classes of stacks of finite type over k modulo the following relations:

1) $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} - \mathcal{Y}]$ where $\mathcal{Y}$ is a closed substack of $\mathcal{X}$,

2) $[\mathcal{X}] = [\mathcal{Y}]$ if there is a radicial and surjective morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over k.

3) $[\mathcal{X}] = [\mathcal{Y} \times \mathbb{A}^r_k]$ where $\mathcal{X} \to \mathcal{Y}$ is a vector bundle of rank r.

The class $[\mathcal{X}]$ in $\mathrm{Mot}(k)$ is called the motivic class of the stack $\mathcal{X}$.

Similarly we can give a ring structure for $\mathrm{Mot}(k)$ by $[\mathcal{X}] \cdot [\mathcal{Y}] := [\mathcal{X} \times_k \mathcal{Y}]$.

Relation between the motivic classes

In special cases, we have the following computations.

Lemma (Ekedahl, 2009)

In $\mathrm{Mot}(k)$ *we have the following results.*
1) $[GL(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$,
2) For a stack $\mathcal{X}$, *we have* $[\mathcal{X}/GL(n)] = [\mathcal{X}]/[GL(n)]$,
3) $[*/GL(n)] = 1/\left((\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})\right)$.

With the above lemma, we find a relation between the motivic classes.

Theorem (Ekedahl, 2009 and L, 2024)

There is a natural group isomorphism :

$$\mathrm{Mot}_{var}(k)[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1} | i > 0] \cong \mathrm{Mot}(k)$$

Stacks of Higgs bundles

Fix a smooth projective geometrically connected curve X over k.

Definition

A Higgs bundle on X is a pair (E, Φ) where E is a vector bundle on X and $\Phi : E \to E \otimes \Omega_X$ is an $\mathcal{O}_X$-linear morphism from E to E "twisted" by the sheaf of differential 1-forms Ω_X. The rank of the pair (E, Φ) is the rank of E, similarly the degree is the degree of E.

Definition

We denote by $\mathcal{H}iggs_{r,d}$ the moduli stack of rank r degree d Higgs bundles on X. This is an Artin stack locally of finite type over k.

Note that the motivic classes of $\mathcal{B}un_{r,d}$ converge, but the motivic classes of $\mathcal{H}iggs_{r,d}$ diverge, that is, have infinite volumes.

Moduli stacks of semistable Higgs bundles

Definition

The Higgs bundle (E, Φ) is called semistable if for any subbundle $F \subset E$ preserved by Φ,

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}.$$

- This is an open condition compatible with field extensions, so we use $\mathcal{H}iggs^{ss}_{r,d} \subset \mathcal{H}iggs_{r,d}$ to denote the open substack of semistable Higgs bundles.
- $\mathcal{H}iggs^{ss}_{r,d}$ is a stack of finite type, thus the corresponding motivic classes converge.

Formulae for semistable Higgs bundles

Theorem (Fedorov-Soibelman-Soibelman, 2018 and L, 2024)

For sufficiently large e, we have $[\mathcal{H}iggs^{ss}_{r,d}] = H_{r,d+er}$*, where* $H_{r,d}$ *is defined via the generating function*

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H_{r,d} w^r z^d = \mathrm{Exp}\left(\sum_{d/r=\tau} B_{r,d} w^r z^d\right),$$

where τ *is any rational number,* Exp *is the plethystic exponential and* $B_{r,d}$ *is a Donaldson-Thomas invariants in* $\overline{\mathrm{Mot}}(k)$.

Thank you!